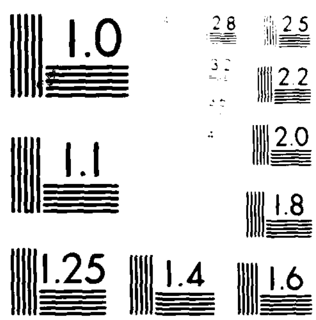
AD-A109 533 SOUTHERN METHODIST UNIV DALLAS TEX DEPT OF STATISTICS F/G 12/1
UTILIZING COMPLEX-VALUES S-ARRAYS IN THE MODELING OF ARMA PROCE--ETC(U)
DEC 81 J D HART, H L GRAY N00014-75-C-0439
UNCLASSIFIED TR-154 NL

1 OF 1
AD A
109533

END
DATE
FILMED
02-82
DTIC




MICROCOPY RESOLUTION TEST CHART

LEVEL

(4)

AD A109533

UTILIZING COMPLEX-VALUES S-ARRAYS IN
THE MODELING OF ARMA PROCESSES

by

Jeffrey D. Hart and Henry L. Gray

Technical Report No. 154
Department of Statistics ONR Contract

December, 1981

Research sponsored by the Office of Naval Research
Contract N00014-75-C-0439



DEPARTMENT OF STATISTICS
Southern Methodist University
Dallas, Texas 75275

# Utilizing Complex-Valued S-Arrays in the Modeling of ARMA Processes


Jeffrey D. Hart
Department of Mathematics
University of Arkansas
Fayetteville, Arkansas 72701

Henry L. Gray
Department of Statistics
Southern Methodist University
Dallas, Texas 75275



Key Words: ARMA processes; Box-Jenkins method; S-array method.



SUMMARY

The problem of estimating the order, (p,q), of an ARMA (p,q) process is considered. An extension of the Gray, Kelley, and McIntire (1978) method of estimating (p,q) is proposed and shown to be particularly useful for processes whose spectra have a certain form. Simulated data is used to illustrate the usefulness of the extension.


## 1. Introduction and Definitions

Gray, Kelley, and McIntire (1978) have illustrated how a certain transform, $S_n(\cdot)$, of the autocorrelation function may be used to estimate p and q from a realization of an ARMA(p,q) process. Their technique hinges on the following two facts.

If $\rho_k$ is the autocorrelation function of an ARMA(p,q) process, then (1)

$$\rho_k - \phi_1\rho_{k-1} - \cdots - \phi_p\rho_{k-p} = 0 \text{ for } k > q \;.$$

Under quite general conditions on the real sequence $\{f_k\}$ , (2)

$$S_n(f_m) = C \text{ for } m \geq m_0 \text{ iff } \{f_m\} \in L(n,\Delta) \text{ for } m > m_0.$$

These two facts imply that p and q can always be determined if the autocorrelation sequence of the process is known. This important "consistency" property is not possessed by the popular Box-Jenkins (1976) method in the case of the mixed process (i.e. $p > 0$ and $q > 0$).

The Gray, et al method of estimating p and q involves the examination of an array, known as the S-array, which contains values of $S_n(\hat{\rho}_k)$, where usually

$$\hat{\rho}_k = \frac{\sum_{t=1}^{N-k} (X_t - \bar{X})(X_{t+k} - \bar{X})}{\sum_{t=1}^{N} (X_t - \bar{X})^2} .$$

A constancy pattern in the S-array consistent with (2) leads to estimates of p and q. Numerous simulation studies and use of the S-array method on real data have indicated that this constancy pattern is sometimes more apparent in an array based on $S_n[(-1)^k\hat{\rho}_k]$. Since $\{\rho_k\} \in L(p,\Delta)$ for $k > q$ if and only if $\{(-1)^k\rho_k\} \in L(p,\Delta)$ for $k > q$, the same theoretical justification exists for using $S_n[(-1)^k\hat{\rho}_k]$ to estimate p and q as for $S_n(\hat{\rho}_k)$. The differing statistical properties of these two transforms will be discussed later.

$S_n(\rho_k)$ and $S_n[(-1)^k\rho_k]$ are members of the following class of transforms of $\rho_k$:

$$\{S_n(e^{2\pi i\omega k}\rho_k): 0 \leq \omega \leq \tfrac{1}{2}\}.$$

The properties of these transforms when $\rho_k$ is the autocorrelation function of an ARMA(p,q) process, and the estimation of p and q by means of $S_n(e^{2\pi i\omega k}\hat{\rho}_k)$, are the subject of the remainder of this paper.

Before proceeding to the next section, the following definitions and notation are given.

A stochastic process $\{X_t\}$, $t = 0, \pm 1, \pm 2, \ldots$, is said to be autoregressive of order p and moving average of order q, or ARMA(p,q), if

$$X_t = \sum_{k=1}^{p} \phi_k X_{t-k} + Z_t - \sum_{k=1}^{q} \theta_k Z_{t-k} \quad \text{for all } t,$$

where the $\phi_k$ and $\theta_k$ are constants and $\{Z_t\}$ is a white noise process with finite variance. If the operator B is defined by $BX_t = X_{t-1}$, then the above may be written as

$$\phi(B)X_t = \theta(B)Z_t, \text{ where}$$

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$$

and

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q .$$

It is well known that $\{X_t\}$ is stationary if and only if all of the roots of $\phi(x) = 0$ lie outside the unit circle.

Let m be an integer and f be a complex-valued function of a real variable. Further, let $f_m = f(m)$,

$$H_n(f_m) = \begin{vmatrix} f_{m-n+1} & f_{m-n+2} & \cdots & f_m \\ f_{m-n+2} & f_{m-n+3} & \cdots & f_{m+1} \\ \vdots & \vdots & & \vdots \\ f_m & f_{m+1} & \cdots & f_{m+n-1} \end{vmatrix},$$

$$H_{n+1}(1; f_m) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_{m-n+1} & f_{m-n+2} & \cdots & f_{m+1} \\ \vdots & \vdots & & \vdots \\ f_m & f_{m+1} & \cdots & f_{m+n-1} \end{vmatrix},$$

where n is a positive integer. Now define

$$S_n(f_m) = \frac{H_{n+1}(1;\, f_m)}{H_n(f_m)}$$

and

$$R_n(f_m) = \begin{cases} \dfrac{H_n(f_m)}{H_n(1; f_m)} & , \quad n = 2,3,\ldots \\ f_m & , \quad n = 1 \end{cases} .$$

The S-array for the function f is the following array of complex numbers:

| m/n | 1 | 2 | ... | k | ... |
|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | |
| -j | $S_1(f_{-j})$ | $S_2(f_{-j})$ | ... | $S_k(f_{-j})$ | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | |
| -2 | $S_1(f_{-2})$ | $S_2(f_{-2})$ | ... | $S_k(f_{-2})$ | ... |
| -1 | $S_1(f_{-1})$ | $S_2(f_{-1})$ | ... | $S_k(f_{-1})$ | ... |
| 0 | $S_1(f_0)$ | $S_2(f_0)$ | ... | $S_k(f_0)$ | ... |
| 1 | $S_1(f_1)$ | $S_2(f_1)$ | ... | $S_k(f_1)$ | ... |
| 2 | $S_1(f_2)$ | $S_2(f_2)$ | ... | $S_k(f_2)$ | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | |
| j | $S_1(f_j)$ | $S_2(f_j)$ | ... | $S_k(f_j)$ | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | |

The following recursion relations, due to Pye and Atchison (1973), are quite helpful in calculating S-arrays. If $S_0(f_m)$ is defined to be 1 for all m, then

$$R_{n+1}(f_m) = R_n(f_{m+1}) \left[ \frac{S_n(f_{m+1})}{S_n(f_m)} - 1 \right]$$

and

$$S_n(f_m) = S_{n-1}(f_{m+1}) \left[ \frac{R_n(f_{m+1})}{R_n(f_m)} - 1 \right] ,$$

where $n = 1,2,\ldots,$ and m is any integer.

A complex-valued sequence $\{f_m\}$ will be said to be an element of $L(n,\Delta)$ for $m > m_0$ if there exists a smallest integer $n > 0$ and a set of $c_i$'s such that

$$f_k + c_1 f_{k-1} + \ldots + c_n f_{k-n} = 0,\ m > m_0 .$$

2. Properties of $S_n(e^{2\pi i\omega k}\rho_k)$

Theorem 1 is now stated in order to given an explicit form for each of the transforms in $\{S_n(e^{2\pi i\omega k}\rho_k): 0 \leq \omega \leq \frac{1}{2}\}$.

Theorem 1 . Let $\{a_k: k=0, \pm 1,\ldots\}$ be any sequence of real or complex numbers. Then for any positive integer $n_0$ and any integer $k_0$, we have

$$S_{n_0}(e^{2\pi i\omega k_0}a_{k_0}) = e^{2\pi i\omega n_0}\frac{\begin{vmatrix} 1 & e^{-2\pi i\omega} & \ldots & e^{-2\pi i\omega n_0} \\ a_{k_0-n_0+1} & a_{k_0-n_0+2} & \ldots & a_{k_0+1} \\ a_{k_0-n_0+2} & a_{k_0-n_0+3} & \ldots & a_{k_0+2} \\ \vdots & \vdots & & \vdots \\ a_{k_0} & a_{k_0+1} & \ldots & a_{k_0+n_0} \end{vmatrix}}{H_{n_0}(a_{k_0})} .$$

Proof: By definition

$$S_{n_0}(e^{2\pi i\omega k_0}a_{k_0}) = \frac{H_{n_0+1}[1;e^{2\pi i\omega k_0}a_{k_0}]}{H_{n_0}(e^{2\pi i\omega k_0}a_{k_0})} .$$

$$H_{n_0+1}[1;e^{2\pi i\omega k_0}a_{k_0}] =$$

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ e^{2\pi i\omega(k_0-n_0+1)}a_{k_0-n_0+1} & e^{2\pi i\omega(k_0-n_0+2)}a_{k_0-n_0+2} & \cdots & e^{2\pi i\omega(k_0+1)}a_{k_0+1} \\ e^{2\pi i\omega(k_0-n_0+2)}a_{k_0-n_0+2} & e^{2\pi i\omega(k_0-n_0+3)}a_{k_0-n_0+3} & \cdots & e^{2\pi i\omega(k_0+2)}a_{k_0+2} \\ \vdots & \vdots & & \vdots \\ e^{2\pi i\omega k_0}a_{k_0} & e^{2\pi i\omega(k_0+1)}a_{k_0+1} & \cdots & e^{2\pi i\omega(k_0+n_0)}a_{k_0+n_0} \end{vmatrix}$$

$$= \exp\{2\pi i\omega[n_0(n_0-1)/2]\} \times$$

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ e^{2\pi i\omega(k_0-n_0+1)}a_{k_0-n_0+1} & e^{2\pi i\omega(k_0-n_0+2)}a_{k_0-n_0+2} & \cdots & e^{2\pi i\omega(k_0+1)}a_{k_0+1} \\ e^{2\pi i\omega(k_0-n_0+1)}a_{k_0-n_0+2} & e^{2\pi i\omega(k_0-n_0+2)}a_{k_0-n_0+3} & \cdots & e^{2\pi i\omega(k_0+1)}a_{k_0+2} \\ \vdots & \vdots & & \vdots \\ e^{2\pi i\omega(k_0-n_0+1)}a_{k_0} & e^{2\pi i\omega(k_0-n_0+2)}a_{k_0+1} & \cdots & e^{2\pi i\omega(k_0+1)}a_{k_0+n_0} \end{vmatrix}$$

$$= e^{2\pi i\omega n_0(k_0+1)} \begin{vmatrix} 1 & e^{-2\pi i\omega} & \cdots & e^{-2\pi i\omega n_0} \\ a_{k_0-n_0+1} & a_{k_0-n_0+2} & \cdots & a_{k_0+1} \\ a_{k_0-n_0+2} & a_{k_0-n_0+3} & \cdots & a_{k_0+2} \\ \vdots & \vdots & & \vdots \\ a_{k_0} & a_{k_0+1} & \cdots & a_{k_0+n_0} \end{vmatrix}$$

The above follows from simple row and column operations. A similar set of operations yields

$$H_{n_0}(e^{2\pi i\omega k_0}a_{k_0}) = e^{2\pi i\omega n_0 k_0}H_{n_0}(a_{k_0}) ,$$

and the result follows.

From Theorem 1 we see that $S_n(e^{2\pi i\omega k}a_k)$ depends on k only through the sequence $\{a_m\}$. This fact will be helpful later when we consider $\{a_m\} = \{\hat{\rho}_m\}$.

As noted previously, the autocorrelation function of an ARMA(p,q) process satisfies the following relationship:

$$\rho_k - \phi_1\rho_{k-1} - \cdots - \phi_p\rho_{k-p} = 0, \; k > q.$$

This relationship, however, is equivalent to

$$e^{2\pi i\omega k}[\rho_k-\phi_1\rho_{k-1}-\cdots-\phi_k\rho_{k-p}] = 0, \; k > q$$

or

$$(e^{2\pi i\omega k}\rho_k)-\phi_1 e^{2\pi i\omega}(e^{2\pi i\omega(k-1)}\rho_{k-1})-\cdots-\phi_p e^{2\pi i\omega p}(e^{2\pi i\omega(k-p)}\rho_{k-p})=0,$$

$$k > q.$$

In other words, for $k > q$, $f_k(\omega) = e^{2\pi i\omega k}\rho_k$, is the solution of a pth order, homogeneous difference equation with complex constant coefficients. If (2) holds for complex sequences, then $f_k(\omega)\varepsilon L(p,\Delta)$ for $k > q$ allows the complex-valued transforms $S_n(e^{2\pi i\omega k}\hat{\rho}_k)$, $0<\omega<\frac{1}{2}$, to be candidates for use in the estimation of p and q. The validity of (2) for complex sequences is established in the following theorem.

<u>Theorem 2</u> Let $\{f_k\}$ be a sequence of complex numbers. Suppose $S_{n_0}(f_m)$ and $R_{n_0}(f_m)$ are defined and $|S_{n_0}(f_m)| > 0$ for $m \geq m_0$.

Then

$$f_m \in L(n_0,\Delta) \text{ for } m > m_0 \text{ iff}$$

$$S_{n_0}(f_m) = C \text{ for } m \geq m_0.$$

Further $C = (-1)^{n_0}(1-a_1-a_2-\ldots-a_{n_0})$ where

$$f_m - a_1 f_{m-1} - \ldots - a_{n_0} f_{m-n_0} = 0,\ m > m_0\ .$$

**Proof:** The proof of this theorem by Gray, Houston, and Morgan (1978) for real $\{f_k\}$ depends only upon properties of determinants and systems of linear equations. These properties hold so long as the elements of $\{f_k\}$ belong to a field. Since the complex number system satisfies the field properties, the result follows.

The basis for the remainder of this work is established in Corollary 1 and Theorem 3.

<u>Corollary 1.</u> Suppose the time series $\{X_t\colon t = 0,\pm 1,\ldots\}$ is a stationary ARMA(p,q) process with autocorrelation $\rho_m$. Suppose $S_{n_0}(f_m(\omega))$ and $R_{n_0}(f_m(\omega))$ are defined, $p > 0$ and $|S_{n_0}(f_m(\omega))| > 0$. Then for some integer $m_0$

$$S_{n_0}(f_m(\omega)) = C_{1,\omega}\ ,\quad m \geq m_0 \text{ and}$$

$$S_{n_0}(f_{m_0-1}(\omega)) \neq C_{1,\omega}$$

if and only if $n_0 = p$ and $m_0 = q$. In addition

$$C_{1,\omega} = (-1)^p(1 - \phi_1 e^{2\pi i\omega} - \phi_2 e^{4\pi i\omega} - \ldots - \phi_p e^{2\pi i\omega p}).$$

**Proof:** Since $\{f_m(\omega)\} \in L(p,\Delta)$ and satisfies the equation

$$f_m(\omega) - (\phi_1 e^{2\pi i\omega}) f_{m-1}(\omega) - \ldots - (\phi_p e^{2\pi i\omega p}) f_{m-p}(\omega) = 0,\ m > q,$$

the results follow immediately from Theorem 2.

Theorem 3 below is the analog of Theorem 3 in Gray, Kelley, and McIntire (1978). Its validity follows from the fact that, since $\rho_k = \rho_{-k}$, we have

$$\rho_{-k} - \phi_1\rho_{-k+1} - \dots - \phi_p\rho_{-k+p} = 0, \; -k < -q.$$

<u>Theorem 3</u> Under the conditions of Theorem 2

$$S_{n_0}(f_m(\omega)) = C_{2,\omega}, \quad m \leq m_1 \text{ and}$$

$$S_{n_0}(f_{m_1+1}(\omega)) \neq C_{2,\omega}$$

iff $n_0 = p$ and $m_1 = -q-1$. In addition,

$$C_{2,\omega} = \frac{-\bar{C}_{1,\omega}e^{2\pi i\omega p}}{\phi_p} \quad .$$

Proof: The proof is completely analogous to the proof of Theorem 3 in Gray, Kelley, and McIntire (1978). Only trivial modifications are required to adjust for the fact that $S_n(f_m(\omega))$ is complex-valued.

Corollary 1 and Theorem 3 show that there is no inherent reason for considering only the real valued transforms $S_n(\hat{\rho}_k)$ and $S_n[(-1)^k\hat{\rho}_k]$ in the estimation of p and q. The question remains, however, as to which transform from the class $\{S_n(e^{2\pi i\omega k}\hat{\rho}_k): 0 \leq \omega \leq \frac{1}{2}\}$ is (in some sense) best for estimating p and q. The best transform must obviously depend on the nature of the process being observed, and thus some method is required to identify this transform for a given process. One immediate clue towards the solution of this problem comes from Corollary 1, which shows that

$$S_p(f_m(\omega)) = (-1)^p(1-\phi_1e^{2\pi i\omega}-\dots-\phi_pe^{2\pi i\omega p}), \; m \geq q. \qquad (3)$$

Since the spectrum, $s_x(\cdot)$, of an ARMA (p,q) process is such that

$$s_x(\omega) \propto \frac{|1 - \theta_1 e^{2\pi i\omega} - \ldots - \theta_q e^{2\pi i\omega q}|^2}{|1 - \phi_1 e^{2\pi i\omega} - \ldots - \phi_p e^{2\pi i\omega p}|^2} \quad (0 \leq \omega \leq \tfrac{1}{2}),$$

it follows that for $m \geq q$, $S_p(f_m(\omega))$ is closely related to $s_x(\omega)$. In fact, if $q = 0$, we have

$$s_x(\omega) \propto \frac{1}{|S_p(f_m(\omega))|^2}, \quad m \geq q.$$

This relationship will be exploited in the next section when the problem of choosing an optimal transform $S_n(e^{2\pi i\omega_0 k}\hat{\rho}_k)$ is formulated.

## 3 Formulation of the Optimal Frequency Problem

Estimating p and q by the S-array method involves examining an array for the presence of a certain constancy pattern. The data at hand supports the estimates $\hat{p}$ and $\hat{q}$ if

$$S_{\hat{p}}(e^{2\pi i\omega k}\hat{\rho}_k) = C_1, \quad k \geq \hat{q} \text{ and}$$

$$S_{\hat{p}}(e^{2\pi i\omega k}\hat{\rho}_k) = C_2, \quad k \leq -\hat{q} - 1 .$$

For a given process, the transform to be used to estimate p and q should be the one which, on the average, makes the correct constancy pattern the most apparent. The ability of $S_n(e^{2\pi i\omega k}\hat{\rho}_k)$ to evidence this constancy depends on two things:

(i) the variability of $S_p(e^{2\pi i\omega k}\hat{\rho}_k)$ for $k \geq q$ and $k \leq -q-1$, and

(ii) the magnitude of the two constants being estimated by $S_p(e^{2\pi i\omega k}\hat{\rho}_k)$ for $k \geq q$ and $k \leq -q - 1$.

These considerations lead to the following definition.

<u>Definition 1</u> Suppose $\{X_t\}$ is a stationary ARMA (p,q) process. Then the frequency $\omega_0$ will be referred to as optimal for the estimation of p and q iff

$$\frac{\mathrm{var}[S_p(e^{2\pi i\omega_0 q}\hat{\rho}_q)]}{|1-\phi_1 e^{2\pi i\omega_0}-\ldots-\phi_p e^{2\pi i\omega_0 p}|^2} \leq \frac{\mathrm{var}[S_p(e^{2\pi i\omega q}\hat{\rho}_q)]}{|1-\phi_1 e^{2\pi i\omega}-\ldots-\phi_p e^{2\pi i\omega p}|^2}$$

for all $\omega \in [0, \frac{1}{2}]$.

In light of (3), our definition of the optimal frequency is consistent with (i) and (ii) above, although we consider the variance of only positive lag S-array values. Note that the quantity considered in the definition is essentially the squared coefficient of variation of the random variable $S_p(e^{2\pi i\omega_0 q}\hat{\rho}_q)$.

In order to obtain an expression for $\mathrm{var}[S_p(e^{2\pi i\omega q}\hat{\rho}_q)]$, recall that, by Theorem 1

$$S_p(e^{2\pi i\omega q}\hat{\rho}_q) = e^{2\pi i\omega p} \frac{\begin{vmatrix} 1 & e^{-2\pi i\omega} & \cdots & e^{-2\pi i\omega p} \\ \hat{\rho}_{q-p+1} & \hat{\rho}_{q-p+2} & \cdots & \hat{\rho}_{q+1} \\ \hat{\rho}_{q-p+2} & \hat{\rho}_{q-p+3} & \cdots & \hat{\rho}_{q+2} \\ \vdots & \vdots & & \vdots \\ \hat{\rho}_q & \hat{\rho}_{q+1} & \cdots & \hat{\rho}_{q+p} \end{vmatrix}}{H_p(\hat{\rho}_q)} .$$

By performing appropriate row and column operations and expanding the numerator by cofactors of the first row, we have

$$S_p(e^{2\pi i\omega q}\hat{\rho}_q) = (-1)^p(1-\hat{\phi}_1^{(p,q)}e^{2\pi i\omega}-\ldots-\hat{\phi}_p^{(p,q)}e^{2\pi i\omega p})$$

where $(\hat{\phi}_1^{(p,q)}, \hat{\phi}_2^{(p,q)},\ldots,\hat{\phi}_p^{(p,q)})$ is the solution of

$$\begin{bmatrix} \hat{\rho}_q & \hat{\rho}_{q-1} & \cdots & \hat{\rho}_{q-p+1} \\ \hat{\rho}_{q+1} & \hat{\rho}_q & \cdots & \hat{\rho}_{q-p+2} \\ \vdots & \vdots & & \vdots \\ \hat{\rho}_{q+p-1} & \hat{\rho}_{q+p-2} & \cdots & \hat{\rho}_q \end{bmatrix} \underline{Y} = \begin{bmatrix} \hat{\rho}_{q+1} \\ \hat{\rho}_{q+2} \\ \vdots \\ \hat{\rho}_{q+p} \end{bmatrix} .$$

By the definition of variance of a complex valued random variable (i.e., $\mathrm{var}(X + iY) = \mathrm{var}(X) + \mathrm{var}(Y)$) we thus have

$$\begin{aligned} \mathrm{var}[S_p(e^{2\pi i\omega q}\hat{\rho}_q)] &= \sum_{k=1}^{p} \mathrm{var}(\hat{\phi}_k^{(p,q)})[\cos^2 2\pi\omega k + \sin^2 2\pi\omega k] \\ &\quad + 2\sum_{j<k}\sum \{(\cos 2\pi\omega j\cos 2\pi\omega k + \sin 2\pi\omega j\sin 2\pi\omega k) \\ &\quad \times \mathrm{cov}(\hat{\phi}_j^{(p,q)}, \hat{\phi}_k^{(p,q)})\} \\ &= \sum_{k=1}^{p} \mathrm{var}(\hat{\phi}_k^{(p,q)}) \\ &\quad + 2\sum_{j<k}\sum \cos[2\pi\omega(k-j)]\mathrm{cov}(\hat{\phi}_j^{(p,q)}, \hat{\phi}_k^{(p,q)}) \\ &= \psi_0 + 2\sum_{k=1}^{p-1} \psi_k \cos 2\pi\omega k \quad , \end{aligned} \tag{4}$$

where $\psi_k = \sum_{j=1}^{p-k} \mathrm{cov}(\hat{\phi}_j^{(p,q)}, \hat{\phi}_{j+k}^{(p,q)})$, $(k = 0,1,\ldots,p-1)$.

Therefore, the frequency which is optimal for the estimation of p and q is the value of $\omega$ which minimizes

$$C(\omega) = \frac{\psi_0 + 2\sum_{k=1}^{p-1} \psi_k \cos 2\pi\omega k}{|1-\phi_1 e^{2\pi i\omega} - \ldots - \phi_p e^{2\pi i\omega p}|^2} \quad . \tag{5}$$

Interestingly, the above expression shows that our criterion for identifying $\omega_0$ is essentially equivalent to the problem of finding the frequency at which the spectrum of an ARMA (p,p-1) process has its minimum.

The above considerations illustrate that for many processes there exists $\omega_0 \neq 0, \frac{1}{2}$ for which $S_n(e^{2\pi i\omega_0 k}\hat{\rho}_k)$ is optimal for the estimation of p and q. We have not, however, addressed the problem of identifying $\omega_0$ given a record of finite length from an ARMA process. A less than optimal, but useful, solution to this problem will be discussed in the next section.

## 4. Estimating the Optimal Frequency $\omega_0$

The dependence of ( 5 ) upon p makes the estimation of $\omega_0$ difficult. Suppose for the moment, however, that for the process under consideration q = 0. Then the quantity $C(\omega)$ is proportional to

$$C^*(\omega) = s_x(\omega)(\psi_0 + 2\sum_{k=1}^{p-1} \psi_k \cos 2\pi\omega k).$$

Experience has shown that for most processes the minimum of $C^*(\omega)$ occurs at about the same frequency as does the minimum of $s_x(\omega)$. This is because $\psi_0 + 2\sum_{k=1}^{p-1} \psi_k \cos 2\pi\omega k$ is almost flat in relation to $s_x(\omega)$. Stated another way, if $\{S_p(e^{2\pi i\omega k}\hat{\rho}_k)\}$ is viewed as a stochastic process with

$$S_p(e^{2\pi i\omega k}\hat{\rho}_k) = S_p(e^{2\pi i\omega k}\rho_k) + e_{p,k}(\omega),$$

then the errors $e_{p,k}(\omega)$ are essentially homoscedastic.

In order to partially substantiate the previous claim, an approximation to $\ell n(NC(\omega))$ has been calculated for two different autoregressive processes and plotted (see Figures 1 and 2) for comparison with

$$\ell n \left\{ \frac{1}{|1-\phi_1 e^{2\pi i\omega} - \ldots - \phi_p e^{2\pi i\omega p}|^2} \right\} .$$

The approximation of $\ell n(NC(\omega))$ uses the Box and Jenkins (1976) approximation for the variance-covariance matrix of

$$\hat{\underline{\phi}}' = (\hat{\phi}_1^{(p,0)}, \hat{\phi}_2^{(p,0)}, \ldots, \hat{\phi}_p^{(p,0)}).$$

Assuming that the noise process $\{a_t\}$ is composed of independent and identically distributed $N(0,\sigma^2)$ random variables, it can be shown that

$$\text{var}(\hat{\underline{\phi}}) \simeq \frac{1}{N}(1 - \underline{\rho}'\underline{\phi})P^{-1},$$

where

$$\underline{\rho}' = (\rho_1, \rho_2, \ldots, \rho_p), \quad \underline{\phi}' = (\phi_1, \phi_2, \ldots, \phi_p),$$

$$P = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{p-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \cdots & 1 \end{bmatrix} ,$$

and N is the sample size.

The processes associated with Figures 1 and 2 are, respectively,

$$(1-.60B)(1-1.0607B+.5625B^2)(1+.4944B+.64B^2)X_t = Z_t \tag{6}$$

and

$$(1-.95B)(1+.95B)(1-1.3435B+.9025B^2)X_t' = Z_t'. \tag{7}$$

As was conjectured, $\ell n(NC(\omega))$ and $-2\ell n(|1-\phi_1 e^{2\pi i\omega}-\ldots-\phi_p e^{2\pi i\omega p}|)$ are minimized at the same value of $\omega$ for process (6) and at approximately the same value for process (7).

The above considerations suggest that, for autoregressive processes, a reasonable estimate of $\omega_0$ would be the value $\hat{\omega}_0$ for which a window spectral estimate $\hat{s}_x(\omega)$ is minimum. It should be pointed out, though, that since $\omega_0$ is of interest only because of its utility in estimating p and q, it is actually not important to have a precise estimate of this parameter. If the array associated with $\hat{\omega}_0$ indicates obvious estimates of p and q, then $\hat{\omega}_0$ has performed its intended function.

The purpose of our discussion to this point has been twofold:

(i) to illustrate theoretically the possible value of complex-valued S-arrays in the estimation of p and q

(ii) to address the problem of estimating a frequency $\omega_0$ for which $S_n(e^{2\pi i\omega_0 k}\hat{\rho}_k)$ is optimal for the estimation of p and q.

Still open for research is a complete solution to the problem in (ii). For the present, however, it is suggested to initially examine the S-array associated with $\hat{\omega}_0$, unless $\hat{\omega}_0$ occurs at a sharp dip in the estimate of the spectrum. Such a sharp dip

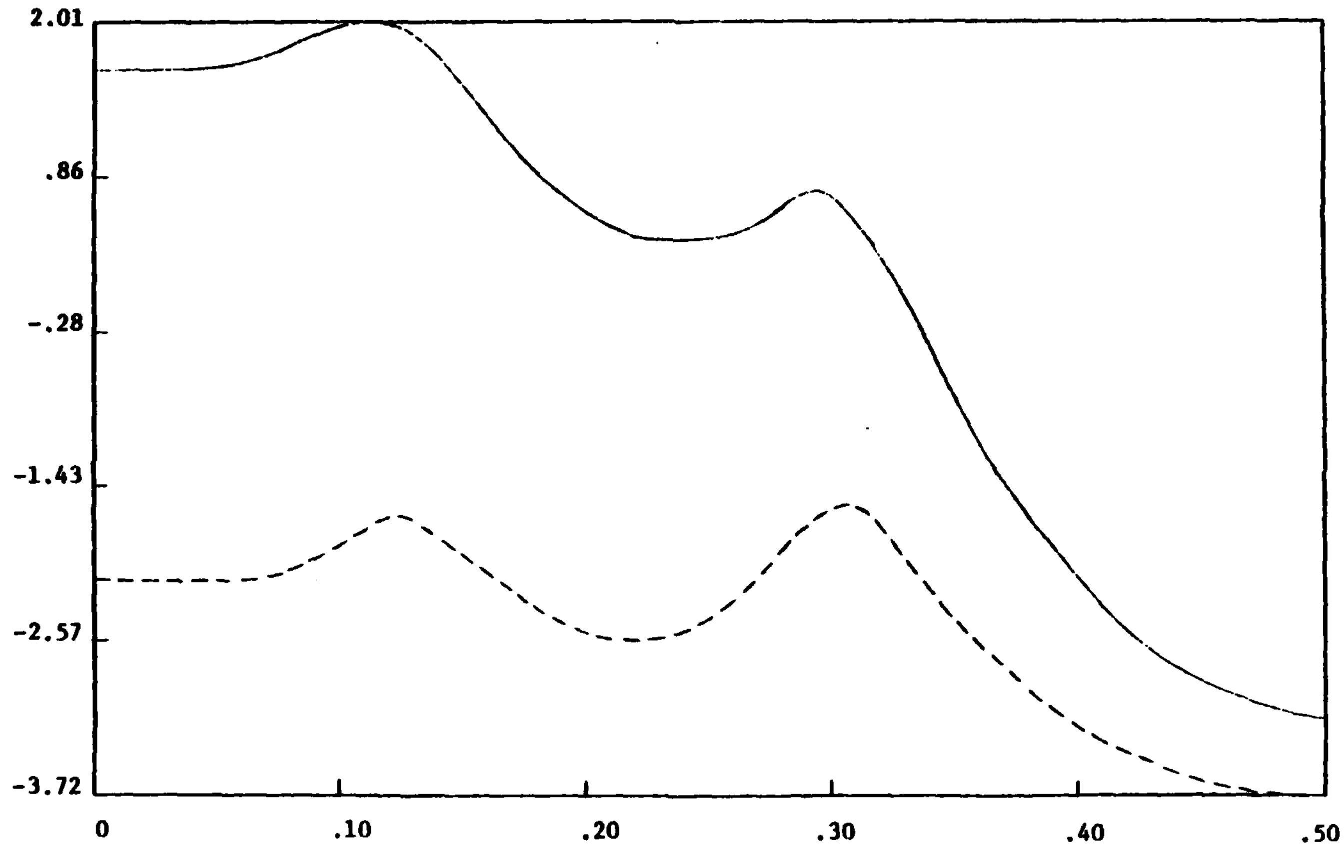



**FIGURE 1**

A Comparison of $\ell n(NC(\omega))$ and $-2\ell n(|1-\phi_1 e^{2\pi i\omega}-\ldots-\phi_p e^{2\pi i\omega p}|)$ for Process (6)

——— $-2\ell n(|1-\phi_1 e^{2\pi i\omega}-\ldots-\phi_p e^{2\pi i\omega p}|)$ - - - - - $\ell n(NC(\omega))$

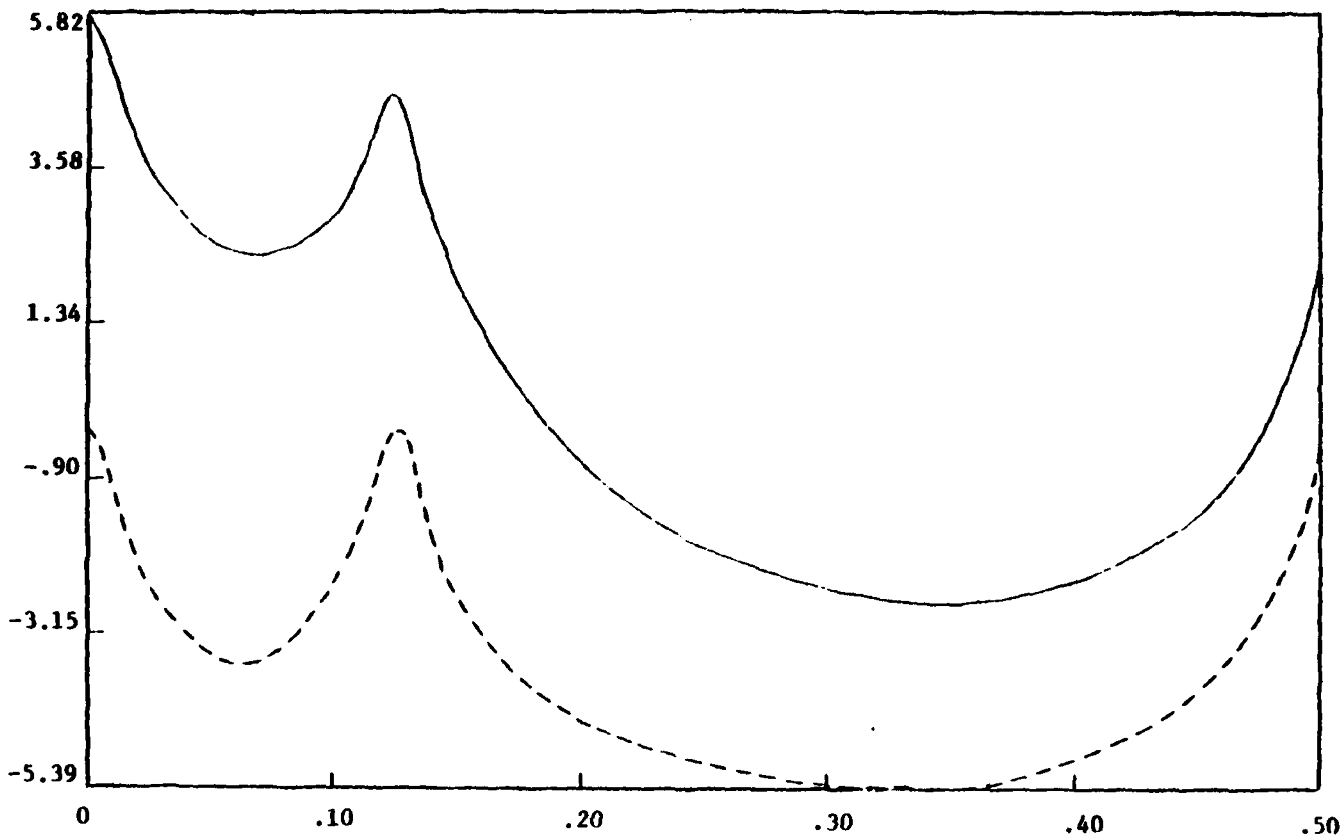



FIGURE 2

A Comparison of $\ell n(NC(\omega))$ and $-2\ell n(|1-\phi_1 e^{2\pi i\omega}-\ldots-\phi_p e^{2\pi i\omega p}|)$ for Process (7)

———— $-2\ell n(|1-\phi_1 e^{2\pi i\omega}-\ldots-\phi_p e^{2\pi i\omega p}|)$ ----- $\ell n(NC(\omega))$

is evidence of a near noninvertible moving average factor, and thus is not indicative of a small value of

$$\frac{1}{|1 - \phi_1 e^{2\pi i\omega} - \ldots - \phi_p e^{2\pi i\omega p}|^2} .$$

If $\hat{\omega}_0$ occurs at a sharp dip, then the S-array associated with the next smallest local minimum should be examined initially. Additional S-arrays may be examined if estimates for p and q are not apparent in the first array. Such an examination of several different arrays may seem prohibitive in terms of computing time, but this is actually not the case due to the recursive algorithm defined in Section 1.

## 5. Examples

The results of Section 4 imply that S-arrays based on frequencies satisfying $0 < \omega < \frac{1}{2}$ will be the most valuable for processes whose spectra have relatively large power at $\omega = 0$ and $\omega = \frac{1}{2}$. Therefore, since the purpose of this section is to illustrate the practical importance of complex-valued S-arrays, the two examples which follow will involve such processes.

In each of the two examples to be considered, five independent realizations of an autoregressive process were generated. The process in Example 1 (whose log-spectrum appears in Figure 3) is

$$(1 - .95B)(1 + .95B)X_t = Z_t , \tag{8}$$

and the process in Example 2 is process (7). Realizations of length $N_1 = 75$ and $N_2 = 50$, respectively, were generated from

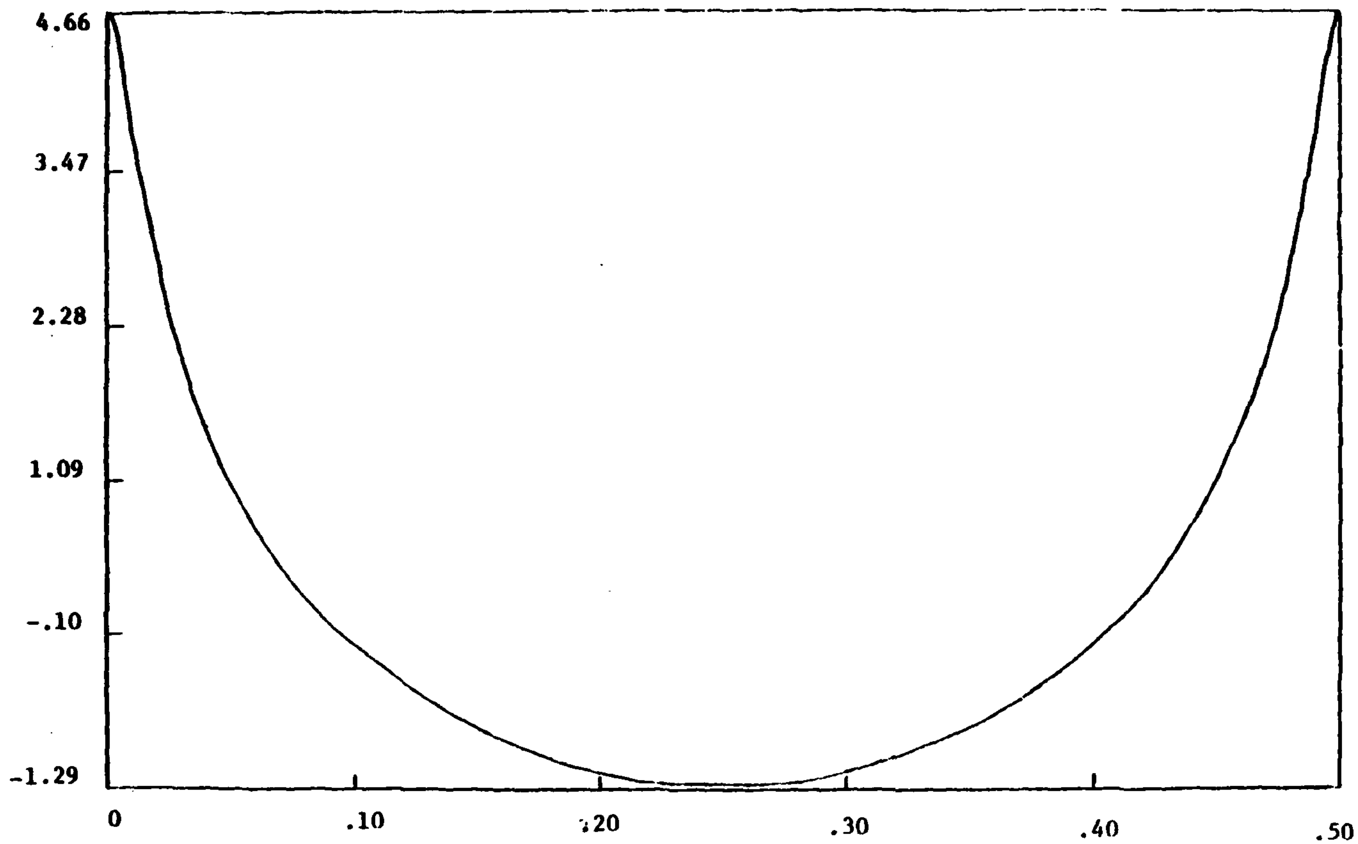



FIGURE 3

The Log-Spectrum of Process (8)

(8) and (7). In each case the values of the noise process $\{Z_t\}$ were obtained by generating random samples from the N(0,1) distribution using IMSL subroutine GGNPM.

For each of the ten realizations an estimate of the spectrum was calculated. The estimates in each case utilized a Parzen window based in Examples 1 and 2, respectively, on 11 and 8 values of the estimated autocorrelation function. The following empirical measures of constancy in the S-array were then computed for each realization in Example j:

$$c_j(\omega) = \frac{\frac{1}{6}\sum_{k=0}^{5} |S_{p_j}(\hat{\rho}_k e^{2\pi i\omega k}) - \bar{S}_{p_j}(\omega)|^2}{|\bar{S}_{p_j}(\omega)|^2},$$

where $p_1 = 2$, $p_2 = 4$, $\bar{S}_{p_j}(\omega) = \frac{1}{6}\sum_{k=0}^{5} S_{p_j}(\hat{\rho}_k e^{2\pi i\omega k})$,

and $\omega = 0, \frac{1}{2}$, and $\hat{\omega}_0$, the frequency at which the estimated spectrum is minimized. Note that $c_j(\omega)$ is a sample analog of the quantity $C(\omega)$.

Table 1 contains the average value of $c_j(0)$, $c_j(\frac{1}{2})$, and $c_j(\hat{\omega}_0)$ over the five realizations of Example j (j = 1,2). In addition, Tables 2 and 3 show the S-arrays associated with $\omega = 0, \frac{1}{2}$, and $\hat{\omega}_0$ for typical realizations in the two examples.

In each example, the average value of $c_j(\hat{\omega}_0)$ is seen to be smaller than the average of either $c_j(0)$ or $c_j(\frac{1}{2})$, and further, $c_j(\hat{\omega}_0)$ was the smallest of the three values for all five realizations in both examples. The numerical evidence in Table 1 favoring S-arrays based on $\hat{\omega}_0$ is presented visually

TABLE 1

AVERAGE VALUES OF $c_1(\omega)$ and $c_2(\omega)$
IN EXAMPLES 1 AND 2

| $j$ | $\overline{c_j(0)}$ | $\overline{c_j(\omega_0)}$ | $\overline{c_j(.5)}$ |
|---|---|---|---|
| 1 | .0878126 | .0005025 | .0361297 |
| 2 | 3.1317649 | .1238482 | .2255245 |

in Tables 2 and 3. Although constancy is apparent in all three arrays in both examples, it is most apparent (for Example j) in the array based on $\hat{\omega}_0$ due to the magnitude of the quantities being estimated in column $p_j$.

## 6. Summary

A refinement of the S-array method of modeling ARMA processes has been introduced in this chapter. In so doing, the theory of complex-valued S-arrays was developed, and the problem of identifying a frequency whose associated S-array is optimal for estimating the order of an ARMA process was formulated. Additionally, two examples involving simulated data were considered in which S-arrays based on estimated optimal frequencies gave clearer determinations of the order of the underlying processes than did real-valued S-arrays.

The true usefulness of the method discussed in this chapter cannot be ascertained until it has been utilized on real data. Constancy patterns in the S-arrays of simulated data tend to appear quite good at all frequencies, and thus the potential worth of complex-valued S-arrays may have been understated in the examples of the previous section. The possibility exists that for certain data, constancy which is virtually hidden in the two real-valued S-arrays is readily apparent in some complex-valued array.

An aspect of the S-array method which has not been discussed here is the information it contains about the

TABLE 2

S-ARRAYS FOR A TYPICAL REALIZATION IN EXAMPLE 1

$\omega = 0$

| m/n | 1 | | 2 | | 3 | | 4 | |
|---|---|---|---|---|---|---|---|---|
| -6 | -.507 | 0.000 | -.145 | 0.000 | .491 | 0.000 | -.393 | 0.000 |
| -5 | 1.286 | 0.000 | -.125 | 0.000 | .046 | 0.000 | .015 | 0.000 |
| -4 | -.510 | 0.000 | -.112 | 0.000 | -.042 | 0.000 | -.094 | 0.000 |
| -3 | 1.261 | 0.000 | -.100 | 0.000 | .148 | 0.000 | -.106 | 0.000 |
| -2 | -.524 | 0.000 | -.105 | 0.000 | .533 | 0.000 | 1.318 | 0.000 |
| -1 | 1.346 | 0.000 | -.084 | 0.000 | -.658 | 0.000 | -1.608 | 0.000 |
| 0 | -.574 | 0.000 | .074 | 0.000 | -.083 | 0.000 | .079 | 0.000 |
| 1 | 1.100 | 0.000 | .093 | 0.000 | -.111 | 0.000 | .222 | 0.000 |
| 2 | -.558 | 0.000 | .089 | 0.000 | -.312 | 0.000 | -.078 | 0.000 |
| 3 | 1.039 | 0.000 | .102 | 0.000 | -.030 | 0.000 | -.053 | 0.000 |
| 4 | -.563 | 0.000 | .111 | 0.000 | .077 | 0.000 | .044 | 0.000 |
| 5 | 1.030 | 0.000 | .131 | 0.000 | -.176 | 0.000 | .199 | 0.000 |

$\omega = .50$

| m/n | 1 | | 2 | | 3 | | 4 | |
|---|---|---|---|---|---|---|---|---|
| -6 | -1.493 | .000 | -.069 | .000 | -.044 | .000 | -.198 | .000 |
| -5 | -3.286 | .000 | -.130 | .000 | .168 | -.000 | -.162 | .000 |
| -4 | -1.490 | .000 | -.098 | .000 | .260 | -.000 | .236 | -.000 |
| -3 | -3.261 | .000 | -.135 | .000 | .093 | -.000 | .162 | -.000 |
| -2 | -1.476 | .000 | -.151 | .000 | -.431 | .000 | -.737 | .000 |
| -1 | -3.346 | .000 | -.210 | .000 | 1.271 | -.000 | -3.103 | .000 |
| 0 | -1.426 | .000 | .183 | .000 | -.160 | -.000 | .152 | .000 |
| 1 | -3.100 | .000 | .134 | .000 | -.090 | -.000 | -.124 | -.000 |
| 2 | -1.442 | .000 | .121 | .000 | .197 | .000 | .118 | .000 |
| 3 | -3.039 | .000 | .089 | .000 | -.186 | -.000 | .133 | .000 |
| 4 | -1.437 | .000 | .116 | .000 | -.280 | -.000 | -.465 | -.000 |
| 5 | -3.030 | .000 | .062 | .000 | -.016 | -.000 | .101 | .000 |

TABLE 2 (con't)

$\omega$ = .298

| m/n | 1 | | 2 | | 3 | | 4 | |
|---|---|---|---|---|---|---|---|---|
| -6 | -1.145 | .471 | 1.927 | .586 | .906 | -4.841 | -.543 | -1.883 |
| -5 | -1.672 | 2.185 | 1.932 | .636 | -2.540 | .265 | 3.085 | .021 |
| -4 | -1.144 | .469 | 1.916 | .615 | -3.373 | 1.439 | 1.912 | 5.973 |
| -3 | -1.665 | 2.161 | 1.919 | .645 | -1.426 | -1.353 | 1.803 | 4.932 |
| -2 | -1.140 | .455 | 1.926 | .656 | 3.149 | -7.722 | 18.318 | -4.375 |
| -1 | -1.690 | 2.242 | 1.930 | .705 | -10.334 | 10.906 | -17.091 | -33.340 |
| 0 | -1.125 | .407 | 1.737 | .438 | -1.873 | -.258 | 1.828 | .177 |
| 1 | -1.617 | 2.007 | 1.739 | .479 | -1.517 | -.845 | .450 | 3.139 |
| 2 | -1.130 | .423 | 1.745 | .488 | .563 | -4.110 | 3.831 | -.102 |
| 3 | -1.600 | 1.949 | 1.746 | .514 | -2.530 | .706 | 3.538 | -.238 |
| 4 | -1.129 | .418 | 1.734 | .496 | -3.593 | 2.311 | 3.315 | 8.209 |
| 5 | -1.597 | 1.941 | 1.737 | .541 | -1.339 | -1.152 | .990 | -.095 |

TABLE 3

S-ARRAYS FOR A TYPICAL REALIZATION IN EXAMPLE 2

$\omega = 0$

| m/n | 1 | | 2 | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| -6 | 8.316 | 0.000 | .897 | 0.000 | -1.065 | 0.000 | -.228 | 0.000 | .259 | 0.000 |
| -5 | .094 | 0.000 | .633 | 0.000 | -2.458 | 0.000 | -.243 | 0.000 | -.691 | 0.000 |
| -4 | -.563 | 0.000 | .703 | 0.000 | -1.232 | 0.000 | -.298 | 0.000 | .837 | 0.000 |
| -3 | -2.791 | 0.000 | .937 | 0.000 | -2.245 | 0.000 | -.200 | 0.000 | .248 | 0.000 |
| -2 | 1.307 | 0.000 | .281 | 0.000 | - .806 | 0.000 | -.214 | 0.000 | .525 | 0.000 |
| -1 | .296 | 0.000 | .583 | 0.000 | -1.156 | 0.000 | -.197 | 0.000 | -4.731 | 0.000 |
| 0 | -.228 | 0.000 | .376 | 0.000 | -.557 | 0.000 | .146 | 0.000 | -.150 | 0.000 |
| 1 | -.566 | 0.000 | .276 | 0.000 | -.999 | 0.000 | .156 | 0.000 | -.250 | 0.000 |
| 2 | -1.558 | 0.000 | .651 | 0.000 | -.744 | 0.000 | .148 | 0.000 | -1.107 | 0.000 |
| 3 | 1.288 | 0.000 | .433 | 0.000 | -1.808 | 0.000 | .212 | 0.000 | -.278 | 0.000 |
| 4 | -.086 | 0.000 | .777 | 0.000 | -1.088 | 0.000 | .179 | 0.000 | -.125 | 0.000 |
| 5 | -.893 | 0.000 | .808 | 0.000 | -1.992 | 0.000 | .169 | 0.000 | -2.798 | 0.000 |

$\omega = .50$

| m/n | 1 | | 2 | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| -6 | -10.316 | .000 | 3.323 | -.000 | -1.396 | -.000 | -.247 | .000 | .210 | -.000 |
| -5 | -2.094 | .000 | 2.997 | -.000 | 3.769 | -.000 | -.227 | .000 | 1.235 | -.000 |
| -4 | -1.437 | .000 | 4.540 | -.000 | -2.035 | .000 | -.311 | .000 | -.193 | .000 |
| -3 | .791 | -.000 | 3.942 | -.000 | 4.510 | -.000 | -.472 | .000 | .423 | -.000 |
| -2 | -3.307 | .000 | 3.756 | -.000 | -1.548 | -.000 | -.501 | .000 | -.232 | .000 |
| -1 | -2.296 | .000 | 4.522 | -.000 | 3.134 | -.000 | -.534 | .000 | 12.043 | -.000 |
| 0 | -1.772 | .000 | 2.913 | -.000 | -1.510 | -.000 | .395 | .000 | -.382 | -.000 |
| 1 | -1.434 | .000 | 3.688 | -.000 | 1.920 | -.000 | .365 | .000 | -.110 | -.000 |
| 2 | -.442 | -.000 | 2.739 | -.000 | -1.494 | -.000 | .349 | .000 | 1.890 | .000 |
| 3 | -3.288 | .000 | 2.800 | -.000 | 2.986 | -.000 | .221 | .000 | -.064 | -.000 |
| 4 | -1.914 | .000 | 3.677 | -.000 | -1.668 | .000 | .168 | .000 | -.224 | -.000 |
| 5 | -1.107 | .000 | 2.994 | -.000 | 2.612 | -.000 | .183 | .000 | 2.275 | .000 |

TABLE 3 (con't)

$\omega = .343$

| m/n | 1 | | 2 | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| -6 | -6.115 | 7.788 | 1.225 | -2.033 | -1.819 | .705 | 4.448 | -2.191 | -4.412 | 1.880 |
| -5 | -1.601 | .915 | 1.325 | -1.736 | -3.415 | -2.622 | 4.454 | -2.218 | -6.951 | 22.342 |
| -4 | -1.240 | .365 | 1.409 | -3.092 | -2.014 | 1.369 | 4.544 | -2.249 | -3.232 | -8.707 |
| -3 | -.016 | -1.497 | 1.253 | -2.577 | -4.624 | -2.777 | 4.547 | -2.056 | -4.450 | 1.256 |
| -2 | -2.266 | 1.928 | 1.550 | -2.387 | -2.371 | 1.008 | 4.578 | -2.071 | -3.373 | -5.569 |
| -1 | -1.712 | 1.083 | 1.465 | -3.071 | -3.798 | -2.240 | 4.568 | -2.016 | -22.550 | 116.096 |
| 0 | -1.424 | .645 | 1.441 | -1.651 | -1.987 | .749 | 3.395 | -1.440 | -3.414 | 1.555 |
| 1 | -1.238 | .362 | 1.547 | -2.327 | -2.681 | -1.737 | 3.386 | -1.400 | -2.959 | -.944 |
| 2 | -.694 | -.467 | 1.297 | -1.510 | -1.667 | .643 | 3.410 | -1.410 | .814 | -20.647 |
| 3 | -2.256 | 1.913 | 1.405 | -1.555 | -2.567 | -2.488 | 3.378 | -1.261 | -3.066 | -.369 |
| 4 | -1.502 | .764 | 1.309 | -2.338 | -1.688 | .883 | 3.441 | -1.275 | -3.645 | 2.171 |
| 5 | -1.059 | .090 | 1.242 | -1.741 | -3.126 | -1.885 | 3.442 | -1.292 | 8.636 | -51.149 |

possible nonstationarity of the observed time series (see Gray, Kelley, and McIntire (1978)). An analog of this important feature of the S-array method is not possessed by automatic order selection techniques such as the AIC criterion of Akaike (1969). The behavior of complex-valued S-arrays under an assumption of nonstationarity is an area for future research. Some unforseen application of these arrays to the nonstationary problem may well exist.

# REFERENCES

Akaike, H. (1969). "Fitting autoregressive models for prediction," Annals of the Institute of Statistical Mathematics, 21, 243-247.

Box, G.E.P. and Jenkins, G.M. (1976). Time Series Analysis: Forecasting and Control revised edition, Holden-Day Inc., San Francisco.

Gray, H. L., Houston, A.G., and Morgan, F.W.(1978). "On G-spectral estimation," in Proceedings of the 1976 Tulsa Symposium on Applied Time Series, New York: Academic Press.

Gray, H. L., Kelley, G.D., and McIntire, D.D. (1978). "A new approach to ARMA modeling," Communications in Statistics, B7, 1-78.

Pye, W.C. and Atchison, T.A. (1973). "An algorithm for the computation of the higher order G-transformation," SIAM Journal on Numerical Analysis, 10, 1-7.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>154 | 2. GOVT ACCESSION NO.<br>AD-A109533 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>Utilizing Complex-Values S-Arrays in the Modeling of ARMA Process | | 5. TYPE OF REPORT & PERIOD COVERED<br>TECHNICAL REPORT<br>6. PERFORMING ORG. REPORT NUMBER<br>154 |
| 7. AUTHOR(s)<br>Jeffrey D. Hart and Henry L. Gray | | 8. CONTRACT OR GRANT NUMBER(s)<br>N00014-75-C-0439 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Southern Methodist University<br>Dallas, Texas 75275 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Office of Naval Research<br>Arlington, VA 22217 | | 12. REPORT DATE<br>December 1981<br>13. NUMBER OF PAGES<br>29 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited. Reproduction in whole or in part is permitted for any purposes of the United States Government.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

ARMA Processes; Box-Jenkins method; S-array method.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The pro-lem of estimating order, (p,q) of an ARMA (p,q) process is considered. An extension of the Gray, Kelley, and McInitre (1978) method of estimating (p,q) is proposed and shown to be particularly useful for processes whose spectra have a certain form. Simulated data is used to illustrate the usefulness of the extension.

ATE

LMED

8